HEAT KERNEL ASYMPTOTICS FOR LAPLACE TYPE OPERATORS AND MATRIX KDV HIERARCHY

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Preliminary version

Abstract. We study the heat kernel asymptotics for the Laplace type differential operators on vector bundles over Riemannian manifolds. In particular this includes the case of the Laplacians acting on differential p-forms. We extend our results obtained earlier for the scalar Laplacian and present closed formulas for all heat invariants associated with these operators. As another application, we present new explicit formulas for the matrix Korteweg-de Vries hierarchy.

1. Introduction and main results

1.1. **Laplace type operators.** Let M be a compact smooth d-dimensional Riemannian manifold without boundary with a metric (g_{ij}) and V be a smooth r-dimensional vector bundle over M. Let $(x_1, \dots, x_d)$ be a system of local coordinates centered at the point $x \in M$ and $(v_1, \dots, v_r)$ be a local frame for V defined near the origin.

Consider a *Laplace type differential operator*, i.e. a second-order elliptic self-adjoint differential operator $D : C^\infty(V) \to C^\infty(V)$ with the scalar leading symbol given by the metric tensor (see [G1]). In the chosen local system it can be written as

$$D = -(\sum_{i,j=1}^{d} g^{ij} I \cdot \partial^2/\partial x_i \partial x_j + \sum_{k=1}^{d} B_k \partial/\partial x_k + C), \tag{1.1.1}$$

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where (g^{ij}) denotes the inverse of the matrix (g_{ij}), I is the identity matrix and $B_k = B_k(x_1, \ldots, x_d)$, $C = C(x_1, \ldots, x_d)$ are some endomorphisms of the bundle V given by $r \times r$ square matrices.

In particular, any Laplacian Δ^p acting on the $\binom{d}{p}$-vector bundle $\Omega^p(M)$ of differential p-forms can be presented in this way (see [Ga]). Another example of a Laplace-type operator is a (1-dimensional) matrix Schrödinger operator (see section 3.1):

$$L = I \cdot \frac{\partial^2}{\partial x^2} + U(x), \tag{1.1.2}$$

where $U(x)$ is a $r \times r$ hermitian matrix.

1.2. **Heat kernel asymptotics.** Let us define the *heat operator* e^{-tD} for $t > 0$ (see [G1]). It is an infinitely smoothing operator from $L^2(V) \to C^\infty(V)$ and it is given by the kernel function $K(x, y, t, D) : V_y \to V_x$ as follows:

$$(e^{-tD} f)(x) = \int\limits_M K(x, y, t, D) f(y) \sqrt{g} dy.$$

where $g = \det(g_{ij})$. The function $K(x, y, t, D)$ is called the *heat kernel* associated with the operator D. It has the following asymptotic expansion on the diagonal as $t \to 0+$ (see [Se], [G1]):

$$K(x, x, t, D) \sim \sum_{n=0}^{\infty} A_n(x, D) t^{n - \frac{d}{2}},$$

where A_n are certain endomorphisms of the fiber V_x.

Consider the spectral decomposition of the self-adjoing operator D into a complete orthonormal basis of eigenfunctions φ_i and eigenvalues λ_i, $0 \le \lambda_1 \le \lambda_2$.... Then

$$K(x, y, t, D) = \sum e^{-\lambda_i t} \varphi_i(x) \otimes \varphi_i(y)$$

Denoting $a_n(x, D) = \mathrm{Tr}(A_n(x, D))$ where Tr is the fiber trace of the endomorphism $A_n(x, D)$, we get

$$\mathrm{Tr}(K(x, x, t, D)) = \sum_i e^{-t\lambda_i} (\varphi_i, \varphi_i)(x) \sim \sum_{n=0}^{\infty} a_n(x, D) t^{n - \frac{d}{2}}.$$

The coefficients $a_n(x, D)$ are the *heat invariants* of the Riemannian manifold M associated with the operator D. They are homogeneous

polynomials of degree $2n$ in the derivatives of the Riemannian metric $\{g^{ij}\}$ at the point x ([G2]).

Computation of heat kernel coefficients is a long-standing problem in spectral geometry (we refer to [P3] for a historical review). In [P3] we calculated heat invariants of Riemannian manifolds for the scalar Laplacian using the Agmon-Kannai (commutator) method (see [P1]). In this paper we present its generalization and extend our formulas for the heat invariants obtained in [P3] to the general Laplace type operators. As another application, we extend the explicit expressions for the scalar Korteweg-de Vries hierarchy obtained in [P2], [P3] to the matrix case using the heat kernel asymptotics of the matrix Schrödinger operator.

The main difficulty in applying the Agmon-Kannai method to matrix-valued operators is that the sum of the orders of two such operators is not necessarily greater than the order of their commutator (cf. (2.2.1)). However, for Laplace type operators this problem in fact does not appear (see section 2.2).

1.3. **Main result.** Denote by $\rho_x : M \to \mathbb{R}$ the distance function on the manifold M: for every $y \in M$ the distance between the points y and x is $\rho_x(y)$. Let us also consider the operator D and its powers as $r\times r$ matrices (cf. (1.1.1)). Let $\mathrm{Tr}(D^{j+n})$ denote the scalar differential operator which is the matrix trace, i.e. the sum of diagonal entries of the operator D^{j+n}.

Theorem 1.3.1. *The endomorphisms $A_n(x, D)$ are equal to*

$$A_n(x,D) = (4\pi)^{-d/2}(-1)^n \sum_{j=0}^{3n} \frac{\binom{3n+d/2}{j+d/2}}{4^j\, j!\,(j+n)!}\, (D^{j+n})(\rho_x(y)^{2j}\cdot I)\Big|_{y=x},$$

and hence the heat invariants $a_n(x, D)$ are given by

$$a_n(x,D) = (4\pi)^{-d/2}(-1)^n \sum_{j=0}^{3n} \frac{\binom{3n+d/2}{j+d/2}}{4^j\, j!\,(j+n)!}\, \mathrm{Tr}(D^{j+n})(\rho_x(y)^{2j})\Big|_{y=x}.$$

The binomial coefficients for d odd are defined by

$$\binom{3n+d/2}{j+d/2} = \frac{\Gamma(3n+1+d/2)}{(3n-j)!\,\Gamma(j+1+d/2)}. \tag{1.3.2}$$

Theorem 1.3.1 is proved in section 3.1.

1.4. **Remarks.** The endomorphisms $A_n(x,D)$ do not depend on the choice of the coordinate system on M and the local frame for V. However, expressions in Theorem 1.3.1 depend on the frame in the fiber V_x. This means that these expressions are subject to certain combinatorial cancellations which should reveal the invariant nature of the endomorphisms $A_n(x,D)$. Note that independence on the choice of the coordinate system is obtained exactly through the study of the combinatorial structure of our formulas (see [P3]).

Another indication of such hidden cancellations (as was noticed by P. Gilkey) is the explicit dependence of the expressions in Theorem 1.3.1 on the dimension d of the manifold M. Indeed, if one writes $A_n(x,D)$ in terms of the curvature tensor and its derivatives, the dimension never appears. Therefore, there should exist a way to rewrite Theorem 1.3.1 in a dimension-independent form as well.

2. The Agmon-Kannai method for Laplace type operators

2.1. **The Agmon-Kannai expansion.** The method for computation of heat invariants developed in [P1], [P3] is based on an asymptotic expansion for resolvent kernels of elliptic operators due to S. Agmon and Y. Kannai ([AK]). Let us briefly review the Agmon-Kannai result.

Definition 2.1.1. (see [AK]). Let $J=(j_1,\dots,j_l)$ be a finite vector with nonnegative integer components and let P, Q be linear operators on a linear space M. The *multiple commutator* $[P,Q;J]$ is defined for all such vectors J by

$$[P,Q;J]=(\operatorname{ad}P)^{j_l}Q\cdots(\operatorname{ad}P)^{j_1}Q,$$

where $(\operatorname{ad}P)Q=[P,Q]$.

Let H be a a self–adjoint elliptic differential operator of order s on a manifold M of dimension $d<s$, and H_0 be the operator obtained by freezing the coefficients of the principal part H' of the operator H at some point $x\in M$: $H_0=H'(x)$. Denote by $R_\lambda(x,y)$ the kernel of the resolvent $R_\lambda=(H-\lambda)^{-1}$, and by $F_\lambda(x,y)$ — the kernel of $F_\lambda=(H_0-\lambda)^{-1}$. The Agmon–Kannai formula in its original form reads as follows ([AK]):

Theorem 2.1.2. *The following asymptotic representation on the diagonal holds for the kernel of the resolvent R_λ as $\lambda \to \infty$:*

$$R_\lambda(x,x) \sim \frac{1}{\sqrt{g}}(F_\lambda(x,x) + \sum_J^{\infty}([H_0 - H, H_0; J]F^{|J|+r+1}(x,x))), \tag{2.1.3}$$

where the sum is taken over all vectors J of length ≥ 1 with nonnegative integer entries.

2.2. **An extension of the Agmon-Kannai theorem.** Denote by $o(P)$ the *order* of the differential operator P (if $P = (P_{ij})$ is a matrix operator then $o(P) = \max_{i,j} o(P_{ij})$). For any two scalar differential operators P, Q we have:

$$o([P,Q]) \leq o(P) + o(Q) - 1 \tag{2.2.1}$$

This simple inequality lies in the basis of the Agmon-Kannai commutator expansions (cf. [AK], Lemma 5.1). Clearly, in the matrix case the relation (2.2.1) does not hold in general, and therefore Theorem 2.1.2 is no longer true. However, under certain assumptions Theorem 2.1.2 can be extended to the matrix case in a straightforward way (see [Kan] for an analogue of this theorem in a much more general situation).

Lemma 2.2.2. *Let H be an $r \times r$ matrix differential operator of order s with a scalar principal part H'. Then the asymptotic expansion (2.1.3) remains valid.*

Proof. Indeed, since H' is a scalar operator, $H_0 = h_0 \cdot I$ is also scalar, $o(H_0) = s$. Let P be an $r \times r$ matrix differential of order p. Consider the commutator $[P, H_0]$. Since H_0 is scalar, the commutator is a matrix operator with the entries $[P_{ij}, h_0]$. For each entry we have: $o([P_{ij}, h_0]) \leq p+s-1$, $i,j = 1,\ldots,r$. Therefore, $o([P, H_0]) \leq p+s-1$. Let us note, that all commutators in the formula (2.1.3) are exactly of the form $[P, H_0]$, where P is some matrix operator. We have shown that inequality (2.2.1) holds for all commutators of this kind, and therefore in this case we may just repeat the proof of Theorem 2.1.2 (see [AK]). This completes the proof of the lemma. ∎

In [P1] we found a concise reformulation of the Agmon–Kannai theorem for scalar operators (Theorem 1.2). Lemma 2.2.2 allows to extend it immediately to the case of matrix operators with a scalar principal part.

Theorem 2.2.3. *Let H be an elliptic differential operator of order s acting on a r-dimensional vector bundle V over a Riemannian manifold M of order $d < s$. The following asymptotic representation on the diagonal holds for the resolvent kernel $R_\lambda(x,y)$ of an elliptic operator H as $\lambda \to \infty$:*

$$R_\lambda(x,x) \sim \frac{1}{\sqrt{g}} \sum_{m=0}^{\infty} X_m F_\lambda^{m+1}(x,x), \tag{2.2.4}$$

where the operators X_m are defined by:

$$X_m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} H^k H_0^{m-k}.$$

Note that the condition $d < s$ can be relaxed in the same way as in [P3] — by taking the derivatives of the resolvent with respect to the spectral parameter λ (see Theorem 2.3.1 in [P3]).

2.3. **Proof of Theorem 1.3.1.** Indeed, consider a normal coordinate system centered at the point $x \in M$. Then $g_{ij}(x) = \delta_{ij}$ and hence the principal part of the operator D with the coefficients frozen at the point x has the form $D_0 = \Delta_0 \cdot I$, where Δ_0 is the scalar Laplacian on R^d and I is the $r \times r$ identity matrix. Due to Theorem 2.2.3 we can repeat the arguments of the proofs of Theorems 1.2.1 and 3.1.1 from [P3] just taking into account multiplication by the matrix I. Finally we obtain the expression for $A_n(x,D)$ identical to the formula (4.2.2) in [P3]. Expressing the distance function in terms of normal coordinates as in [P3] we obtain the formulas for the endomorphisms $A_n(x,D)$. Taking the traces of $A_n(x,D)$ we get the heat invariants $a_n(x,D)$ and this completes the proof of the theorem. □

3. Matrix KdV hierarchy

3.1. **Heat kernel asymptotics for Schrödinger operator.** Consider the 1-dimensional matrix Schrödinger operator L (see (1.1.2)) with a potential U which is an hermitian $r \times r$ matrix. Its heat kernel $H(t,x,y)$ is the fundamental solution of the heat equation

$$\left(\frac{\partial}{\partial t} - L\right) f = 0.$$

It has the following asymptotic representation on the diagonal as $t \to 0+$:

$$H(t,x,x) \sim \frac{1}{\sqrt{4\pi t}} \sum_{n=0}^{\infty} h_n[U] t^n,$$

where $h_n[U]$ are some polynomials in the matrix $U(x)$ and its derivatives.

The matrix KdV hierarchy is defined by (see [AvSc1]):

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x} G_n[U], \tag{3.1.1}$$

where

$$G_n[U] = \frac{(2n)!}{2 \cdot n!} h_n[U], \quad n \in \mathbb{N}.$$

Set $U_0 = U$, $U_n = \partial^n U / \partial x^n$, $n \in \mathbb{N}$, where U_n, $n \geq 0$ are formal variables. The sequence of polynomials $G_n[U] = G_n[U_0, U_1, U_2, \dots]$ starts with (see [AvSc2]):

$$G_1[U] = U_0, \quad G_2[U] = U_2 + 3U_0^2,$$
$$G_3[u] = U_4 + 5U_0U_2 + 5U_2U_0 + 5U_1^2 + 10U_0^3, \ \dots$$

3.2. **Computation of matrix KdV hierarchy.** In [P2], [P3] we presented explicit formulas for the scalar Korteweg-de Vries hierarchy using heat invariants of the scalar 1-dimensional Schrödinger operator (we refer to [P2] for the history of this question). Theorem 1.3.1 allows to extend our results to the matrix KdV hierarchy as well.

For other approaches to explicit computations of the matrix KdV hierarchy see [AvSc1], [AvSc2].

Theorem 3.2.1. *The KdV hierarchy is given by:*

$$G_n[U] = \frac{(2n)!}{2 \cdot n!} \sum_{j=0}^{n} \binom{n+\frac{1}{2}}{j+\frac{1}{2}} \frac{(-1)^j}{4^j \, j! \, (j+n)!} P_{nj}[U],$$

where the polynomial $P_{nj}[U]$ is obtained from $L^{j+n}(x^{2j})|_{x=0}$ by a formal change of variables: $U_i(0) \to U_i$, $i = 0,..,2n+2j-2$.

This expression can be completely expanded due to a formula for the powers of the Schrödinger operator ([Rid]) which remains valid in the matrix case as well (note that now U_i are matrices and hence do not commute).

Theorem 3.2.2. *The polynomials $G_n[U]$, $n \in \mathbb{N}$ are equal to:*

$$G_n[U] = \frac{(2n)!}{2 \cdot n!} \sum_{j=0}^{n} \binom{n+\frac{1}{2}}{j+\frac{1}{2}} \frac{(-1)^j (2j)!}{4^j\, j!\, (j+n)!} \sum_{p=1}^{j+n} \sum_{\substack{k_1,\ldots,k_p \\ k_1+\cdots+k_p=2(n-p)}} C_{k_1,\ldots,k_p} U_{k_1} \cdots U_{k_p},$$

where

$$C_{k_1,\ldots,k_p} = \sum_{\substack{0 \le l_0 \le l_1 \le \cdots \le l_{p-1} = j+n-p \\ 2l_i \ge k_1+\cdots+k_{i+1},\ i=0,\ldots,p-1.}} \binom{2l_0}{k_1} \binom{2l_1 - k_1}{k_2} \cdots \binom{2l_{p-1} - k_1 - \cdots - k_{p-1}}{k_p}.$$

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References

[AK] S. Agmon, Y. Kannai, On the asymptotic behavior of spectral functions and resolvent kernels of elliptic operators, Israel J. Math. 5 (1967), 1-30.

[AvSc1] I.G. Avramidi, R. Schimming, A new explicit expression for the Korteweg - de Vries hierarchy, solv-int/9710009, (1997), 1-17.

[AvSc2] I.G. Avramidi, R. Schimming, Heat kernel coefficients for the matrix Schrödinger operator, J. Math. Phys., 36 (1995), no. 9, 5042-5054.

[Ga] M.P. Gaffney, Asymptotic distributions associated with the Laplacian on forms, Comm. Pure Appl. Math., vol. 11 (1958), 535-545.

[G1] P. Gilkey, The spectral geometry of a Riemannian manifold, J. Diff. Geom. 10 (1975), 601-618.

[G2] P. Gilkey, The index theorem and the heat equation, Math. Lect.Series, Publish or Perish, 1974.

[G3] P. Gilkey, Invariance theory, the heat equation and the Atiyah-Singer index theorem, Mathematics Lecture Series, 11, Publish or Perish, 1984.

[Kan] Y. Kannai, On the asymptotic behavior of resolvent kernels, spectral functions and eigenvalues of semi-elliptic systems, Ann. Scuola Norm. Sup. Pisa (3), 23 (1969), 563-634.

[P1] I. Polterovich, A commutator method for computation of heat invariants, Indag. Math., N.S., (2000) 11(1) 139-149.

[P2] I. Polterovich, From Agmon-Kannai expansion to Korteweg-de Vries hierarchy, Lett. Math. Phys. 49 (1999), 71-77.

[P3] I. Polterovich, Heat invariants of Riemannian manifolds, to appear in Israel Journal of Mathematics.

[Rid] S.Z. Rida, Explicit formulae for the powers of a Schrödinger-like ordinary differential operator, J. Phys. A: Math. Gen. 31 (1998), 5577-5583.

[Se] R. Seeley, Complex powers of an elliptic operator, Proc. Symp. Pure Math. 10 (1967), 288-307.

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